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A NEW APPROXIMATION OPERATOR OF THE ARATO-RENYI TYPE

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ABSTRACT

In this paper a new approximation operator is introduced and its properties are studied. Special cases of this operator are the well-known Szàsz power-series approximation operator and its generalization by D. Leviatan. The behaviour of the new approximation operator at points of continuity and discontinuity is investigated by using probabilistic tools as the Chebishev inequality and Liapounov's central limit theorem. Such probabilistic methods of proof simplify the proofs and give better understanding of the approximation mechanism.

0. Introduction

In paper we introduce a new approximation operator of the Arato-Renyi type. This new operator generalizes the well known Szàsz power series and an operator of D. Leviatan (introduced in [4]). The motivation to define and investigate such an operator came from the interesting paper of M. Arato and A. Renyi [1].

Using probabilistic tools such as the Chebishev inequality, a variant of Liapounov's central limit theorem and a simple form of Kolmogorov's three series theorem we analyse the behavior of our operator at continuity points of the approximated function as well as at discontinuity points.

The next section gives the theorems while their proofs are given in Section 2.

1. Main results

Let $X_j\,(j \geqq 1)$ be non-negative independent random variables with finite means m_j and finite variances b_j satisfying

$$\sum_{j=1}^{\infty} b_j < \infty . \tag{1.1}$$

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As is well-known (see [6, theor. 2, p. 423]) for each $k = 1, 2, \cdots$ the series $\Sigma_{j=k}^{\infty}(X_j - m_j)$ converges almost surely to an almost surely finite random variable, to be denoted by S_k, having finite variance $\Sigma_{j=k}^{\infty} b_j$.

Thus we may define for $-\infty \leqq x \leqq +\infty$

$$(1.2)\qquad \begin{cases} P_0(x) \equiv P_0(\{X_j\}, x) \equiv \Pr(S_1 \leqq x) \\ P_k(x) \equiv P_k(\{X_j\}_x) \equiv \Pr\left(S_{k+1} - m_k \leqq x + \sum_{j=1}^{k-1} m_j < S_k\right) \qquad (k \geqq 1). \end{cases}$$

The first approximation operator defined for every function f on $[0, \infty]$ is

$$(1.3)\qquad P_u(f, x) \equiv P_u(\{X_j\}, f, x) \equiv \sum_{k=0}^{\infty} f(u^{-1}\rho_k) P_k(-\log xu) \qquad (u > 0)$$

where

$$(1.4)\qquad \rho_0 \equiv \rho_0(\{X_j\}) = 1, \ \rho_k \equiv \rho_k(\{X_j\}) \equiv \exp\left(\sum_{j=1}^{k} m_j\right) \qquad (k \geqq 1).$$

The properties of this operator for points of continuity are established in Theorem 1 while Theorem 2 establishes its properties at points of discontinuity.

THEOREM 1. *Let X_j $(j \geqq 1)$ be non-negative independent random variables with corresponding finite means m_j and finite variances b_j. Suppose* (1.1) *is satisfied,*

$$(1.5)\qquad \sum_{j=1}^{\infty} m_j = +\infty$$

and

$$(1.6)\qquad \lim_{j \to \infty} m_j = 0.$$

Then:

(i) *If $f(x)$ is continuous for $0 \leqq x \leqq +\infty$ then we have uniformly in $0 \leqq x \leqq +\infty$*

$$(1.7)\qquad \delta_{x,+\infty} f(+\infty) + \lim_{u \to +\infty} P_u(f, x) = f(x),$$

where $\delta_{u,v} = 0$ for $u \neq v$ and $\delta_{uu} = 1$.

(ii) *If $f(x)$ is bounded for $0 \leqq x < +\infty$ and continuous at $x = x_0$, $0 \leqq x_0 < +\infty$, then*

$$(1.8)\qquad \lim_{u \to +\infty} P_u(f, x_0) = f(x_0).$$

Suppose X_j $(j \geqq 1)$ is a sequence of non-negative independent random variables with corresponding finite means m_j and finite variances b_j satisfying (1.1), (1.5) and (1.6). Let $\{n_j\}_{j\geqq 1}$ be a sequence of integers such that $1 = n_1 < n_2 < \cdots < n_j \to +\infty$ and $\lim_{j\to\infty} \sum_{r=n_j}^{n_{j+1}-1} m_r = 0$. Then the sequence Y_j $(j \geqq 1)$, $Y_j = \sum_{r=n_j}^{n_{j+1}-1} X_j$ is a sequence of non-negative independent random variables satisfying (1.1), (1.5) and (1.6). Also

$$P_0(\{Y_m\}), x) = P_0(\{X_m\}, x), \quad P_j(\{Y_m\}, x) = \sum_{r=n_j}^{n_{j+1}-1} P_r(\{X_j\}, x) \qquad (j \geqq 1)$$

$$\rho_0(\{Y_m\}) = 1 \qquad \rho_j(\{Y_m\}) = \rho_{n_{j+1}-1}(\{X_m\}) \qquad (j \geqq 1).$$

Thus for each suitable sequence $\{n_j\}_{j\geqq 1}$ we obtain a new approximation operator $P_u(\{Y_m\}, f, x)$ to which Theorem 1 applies and whose coefficients are related in a simple way to those of $P_u(\{X_m\}, f, x)$.

THEOREM 2. *Let $X_j (j \geqq 1)$ be non-negative independent random variables with means m_j and nonzero variances b_j satisfying* (1.1), (1.5) *and* (1.6). *Moreover, if for some x_0, $0 < x_0 < +\infty$, one has*

$$\left(\sum_{j=n(u)+1}^{\infty} E\,|X_j - m_j|^{2+\delta}\right)^{1/(2+\delta)} = o\left(\sum_{j=n(u)+1}^{\infty} b_j\right)^{1/2} \qquad (u \to \infty) \tag{1.9}$$

for some $\delta > 0$,

$$m_{n(u)+1} = o\left(\sum_{j=n(u)+1}^{\infty} b_j\right)^{1/2} \qquad (u \to \infty) \tag{1.10}$$

where $n(u) \equiv n(u, x_0) = \max\{j : u^{-1}\rho_j \leqq x_0\}$. Then

$$\tfrac{1}{2}(l^+(x_0) + l^-(x_0)) \leqq \varliminf_{u\to\infty} P_u(f, x_0) \leqq \varlimsup_{u\to\infty} P_u(f, x_0) \leqq \tfrac{1}{2}(L^+(x_0) + L^-(x_0)) \tag{1.11}$$

where $L^+(x_0)$, $L^-(x_0)$, $l^+(x_0)$, $l^-(x_0)$ are, respectively, the right and left upper limits and the right and left lower limits of $f(x)$ at x_0. In particular if $f(x_0)$ and $f(x_0-)$ exist we have

$$\lim_{u\to\infty} P_u(f, x_0) = \tfrac{1}{2}(f(x_0+) + f(x_0-)). \tag{1.12}$$

In the following theorem we obtain for non-negative independent random variables X_j which are exponentially distributed with means $m_j = a_j^{-1}$ an explicit form for the functions $P_k(\cdot)$ used in defining the approximation operator $P_u(f, x)$.

For a random variable X denote by $F_X(\cdot)$, $\varphi_X(\cdot)$ and $f_X(\cdot)$ the corresponding distribution function, characteristic function and the probability density function (if it exists).

THEOREM 3. *Suppose* $a_j > 0\,(j \geqq 1)$,

$$\sum_{j=1}^{\infty} a_j^{-1} = +\infty \quad \text{and} \quad \sum_{j=1}^{\infty} a_j^{-2} < +\infty. \tag{1.13}$$

Let X_j $(j \geqq 1)$ *be independent exponential random variables with means* a_j^{-1}. *Denote for* $k = 1, 2, \cdots, f_k(\cdot) \equiv f_{S_k}(\cdot)$ *and* $F_k(\cdot) \equiv F_{S_k}(\cdot)$. *Then*

(i) *For* $k = 1, 2, \cdots, f_k(\cdot) \in C^{\infty}(-\infty, +\infty)$ *and*

$$F_1(t) = \int_{-\infty}^{t} f_1(s)\,ds. \tag{1.14}$$

(ii) *For* $k = 1, 2, \cdots$ *and* $-\infty < x < +\infty$

$$f_{k+1}(x) = \left(1 + \frac{D}{a_k}\right) f_k(x - a_k^{-1}) = \left[\prod_{j=1}^{k}\left(1 - \frac{D}{a_j}\right)\right] f_1\left(x - \sum_{r=1}^{k} a_r^{-1}\right) \tag{1.15}$$

where D *denotes differentiation with respect to* x.

(iii) *For* $k = 1, 2, \cdots$ *and* $-\infty < x < +\infty$

(1.16)

$$\Pr\left(S_{k+1} - \frac{1}{a_k} \leqq x < S_k\right) = \frac{1}{a_k}\left[\prod_{j=1}^{k-1}\left(1 + \frac{D}{a_j}\right)\right] f_1(x) = \frac{1}{a_k}\left[D\prod_{j=1}^{k-1}\left(1 - \frac{D}{a_j}\right)\right] F_1(x).$$

(iv) *For*

$$E(s) \equiv \prod_{j=1}^{\infty}\left(1 - \frac{s}{a_j}\right) e^{s/a_j} \tag{1.17}$$

which converges for all complex s *we have*

$$f_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itx}}{E(ix)}\,dx \tag{1.18}$$

and for all complex s *satisfying* $\operatorname{Re} s < \min_{j \geq 1} a_j$ *one has*

$$\frac{1}{E(s)} = \int_{-\infty}^{+\infty} e^{-st} f_1(t)\,dt. \tag{1.19}$$

(v) *In this case denote the operator* $P_u(f, x)$ *by* $S_u(f, x)$; *then*

(1.20)

$$S_u(f, x) = f(u^{-1}) F_1(-\log xu) + \sum_{k=1}^{\infty} f(u^{-1}\xi_k) a_k^{-1} \prod_{j=1}^{k-1}\left(1 + \frac{D}{a_j}\right) f_1(-\log xu + \log \xi_{k-1})$$

where

$$\xi_k = \exp\left(\sum_{j=1}^{k} a_j^{-1}\right). \tag{1.21}$$

AN EXAMPLE. If we take in the last theorem $a_n = n$ $(n \geqq 1)$ (which clearly satisfy our conditions on the a_n) we get the following variant of the Szàsz approximation operator

$$S_u(f, x) = \sum_{k=0}^{\infty} f\left(\frac{1}{u}\exp\sum_{j=1}^{k}\frac{1}{j}\right)\frac{1}{k!}\exp(-xue^{-\gamma})(xue^{-\gamma})^k \tag{1.22}$$

where γ is Euler's constant i.e.,

$$\gamma = \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j} - \log n\right).$$

Indeed using the inversion formula for Laplace transforms along with the residue theorem and standard estimates for integrals along circular curves we find

$$F_1(-\log x) = \exp(-xe^{-\gamma}). \tag{1.23}$$

This equation together with (1.16) yields (1.22).

The next two theorems state some approximation properties of the operator $S_u(f, x)$.

By applying Theorem 3 to Theorem 1 we obtain

THEOREM 4. *Suppose that* $a_j > 0$ $(j \geqq 1)$ *satisfy* (1.13).
Then:

(i) *If* $f(x)$ *is continuous for* $0 \leqq x \leqq \infty$ *we have*

$$\delta_{x,\infty} f(+\infty) + \lim_{u\to\infty} S_u(f, x) = f(x) \text{ uniformly for } 0 \leqq x \leqq \infty. \tag{1.24}$$

(ii) *If* $f(x)$ *is bounded for* $0 \leqq x \leqq \infty$ *and continuous at* $x = x_0$ *then*

$$\lim_{u\to\infty} S_u(f, x_0) = f(x_0). \tag{1.25}$$

Result (ii) was first proved by D. Leviatan [4].

THEOREM 5. *Suppose* a_n $(n \geqq 1)$ *satisfy the assumptions of Theorem* 3. *If in addition to the previous assumption one has for some* x_o, $0 < x_0 < +\infty$

$$\left(\sum_{j=k(u)+1}^{\infty} a_j^{-3}\right)^{1/3} = o\left(\left(\sum_{j=k(u)+1}^{\infty} a_j^{-2}\right)^{1/2}\right) \qquad (u \to \infty) \tag{1.26}$$

where $k(u) \equiv k(u, x_0) \equiv \max\{j : u^{-1}\xi_j \leqq x_0\}$, *then for each bounded function* $f(x)$ *in* $[0, \infty]$ *we have*

$$\tfrac{1}{2}(l^+(x_0) + l^-(x_0)) \leqq \varliminf_{u\to\infty} S_u(f, x) \leqq \varlimsup_{u\to\infty} S_u(f, x) \leqq \tfrac{1}{2}(L^+(x_0) + L^-(x_0)). \tag{1.27}$$

In particular if x_0 *is a point of discontinuity of the first type for* $f(x)$ *one gets*

$$\lim_{u\to\infty} S_u(f, x) = 1/2\{f(x_0+) + f(x_0-)\}. \tag{1.28}$$

An analogous result for the generalized Bernstein polynomials was obtained by A. Jakimovski [3].

2. Proofs

In proving Theorem 1 we rely on the following lemma.

LEMMA 1. *Let* X_j $(j \geqq 1)$ *be non-negative independent random variables which satisfy* (1.1), (1.5) *and* (1.6). *Then*

$$\sum_{k=0}^{\infty} P_k(-\infty) = 0 \ \textit{and} \ \sum_{k=0}^{\infty} P_k(x) = 1 \ \textit{for} \ -\infty < x \leqq +\infty. \tag{2.1}$$

The authors wish to thank Professor D. Leviatan for pointing out that Lemma 1 requires a proof.

PROOF. We have $P_0(+\infty) = 1$ *and* $P_k(+\infty) = 0$ for $k \geqq 1$. So (2.1) is true for $x = +\infty$. Also $P_k(-\infty) = 0$ for $k \geqq 0$ so (2.1) is true also for $x = -\infty$. For $-\infty < x < +\infty$ we have, since X_j $(j \geqq 1)$ is non-negative,

$$\sum_{k=0}^{n} P_k(x) = \Pr\left(\Omega - \left(S_{n+1} > x + \sum_{j=1}^{n} m_j\right)\right) = 1 - \Pr\left(S_{n+1} > x + \sum_{j=1}^{n} m_j\right), \tag{2.2}$$

and by the Chebishev inequality we get

$$\Pr\left(S_{n+1} > x + \sum_{j=1}^{n} m_j\right) \leqq \left(x + \sum_{j=1}^{n} m_j\right)^{-2} \sum_{j=n+1}^{\infty} b_j \to 0, \ n \to \infty.$$

Therefore for $-\infty < x < +\infty$

$$\sum_{k=0}^{\infty} P_k(x) = \lim_{n\to\infty} \sum_{k=0}^{n} P_k(x) = 1.$$

Q.E.D.

PROOF OF THEOREM 1. We prove part (i) of the theorem. Part (ii) is proved in a similar way. Given $\varepsilon>0$ there exists some $\delta=\delta(\varepsilon)>0$ such that $|f(y)-f(x)|<\varepsilon$ for $|y-x|<\delta$, $0\leqq x$, $y<+\infty$ and $|f(y)-f(+\infty)|<\varepsilon$ for $y>\delta^{-1}$.

By Lemma 1 we have

$$\delta_{x,+\infty}f(+\infty)+P_u(f,x)-f(x)=\begin{cases}0 & \text{for } x=+\infty\\ I_1(u)+I_2(u)+I_3(u) & \text{for } 0<x<+\infty\\ f\left(\frac{1}{u}\right)-f(0) & \text{for } x=0\end{cases}\tag{2.3}$$

where for $0<x<+\infty$

$$I_1(u)\equiv\sum_{\substack{k\\ \{k:|u^{-1}\rho_k-x|<\delta\}}}[f(u^{-1}\rho_k)-f(x)]P_k(-\log xu),$$

$$I_2(u)\equiv\sum_{\substack{k\\ \{k:u^{-1}\rho_k-x\geqq\delta\}}}[f(u^{-1}\rho_k)-f(x)]P_k(-\log xu),$$

$$I_3(u)\equiv\sum_{\substack{k\\ \{k:u^{-1}\rho_k-x\leqq-\delta\}}}[f(u^{-1}\rho_k)-f(x)]P_k(-\log xu).$$

Obviously

$$|I_1(u)|\leqq\varepsilon \qquad \text{for } 0<x<+\infty.\tag{2.4}$$

We estimate $I_2(u)$ for $0<x\leqq\delta^{-1}$ and for $+\infty>x>\delta^{-1}$ separately. The function $k_1(x,u)$ defined by

$$k_1\equiv k_1(x,u)\equiv\min\{k:u^{-1}\rho_k-x\geqq\delta\}$$

is an increasing function in x and in u separately for $0\leqq x$ and $u>0$. Now for $\delta^{-1}<x<+\infty$ we have

$$\begin{aligned}|I_2(u)|&\leqq\sum_{k=k_1}^{\infty}|f(u^{-1}\rho_k)-f(x)|P_k(-\log xu)\\ &\leqq\sum_{k=k_1}^{\infty}[|f(u^{-1}\rho_k)-f(+\infty)|+|f(x)-f(+\infty)|]P_k(-\log xu)\\ &\leqq 2\varepsilon\sum_{k=0}^{\infty}P_k(-\log xu)\end{aligned}\tag{2.5}$$

(and by (2.1))

$$\leqq 2\varepsilon \qquad \text{for } \delta^{-1}<x<+\infty.$$

We have $|f(x)| \leqq M < +\infty$ for $0 \leqq x \leqq +\infty$. Now for $0 < x \leqq \delta^{-1}$ we get

$$(2.6) \quad |I_2(u)| \leqq 2M \cdot \sum_{k=k_1}^{\infty} P_k(-\log xu)$$

$$= 2M \cdot \lim_{p \to +\infty} \sum_{k=k_1}^{p} \Pr(S_{k+1} - m_k \leqq -\log xu + \log \rho_{k-1} < S_k)$$

(and by (2.2))

$$= 2M \cdot \{\Pr(S_{k_1} > -\log xu + \log \rho_{k_1-1})$$

$$- \lim_{p \to +\infty} \Pr(S_{p+1} > -\log xu + \log \rho_p)\}$$

$$\leqq 2M \cdot \Pr(S_{k_1} > -\log xu + \log \rho_{k_1-1})$$

(and since $u^{-1}\rho_k - x \geqq \delta$ implies $\log \rho_{k_1-1} - \log xu \geqq \log(1 + \delta x^{-1}) - m_{k_1}$)

$$\leqq 2M \Pr(S_{k_1} > \log(1 + \delta x^{-1}) - m_{k_1}), \qquad \text{for } 0 < x < \delta^{-1}.$$

By (1.6) we can find $K(\varepsilon)$ such that

$$(2.7) \qquad m_k < \frac{1}{2}\log(1 + \delta^2) \qquad \text{for } k > K(\varepsilon).$$

Since $k_1(x, u)$ is increasing for $0 \leqq x < +\infty$ and $u > 0$ and $k_1(0, u) \to +\infty$ as $u \to +\infty$ we get that $u > u_0(\varepsilon)$ implies $k_1(0, u) > K(\varepsilon)$ and in particular, for $0 \leqq x \leqq \delta^{-1}$ and $u > u_0(\varepsilon)$, $k_1(x, u) \geqq k_1(0, u) > K(\varepsilon)$. Therefore by (2.7)

(2.8)

$$m_{k_1(x,u)} < \frac{1}{2}\log(1 + \delta^2) \leqq \frac{1}{2}\log(1 + \delta x^{-1}) \qquad \text{for } 0 < x \leqq \delta^{-1} \text{ and } u > u_0(\varepsilon).$$

Hence, for $0 < x \leqq \delta^{-1}$ and $u > u_0(\varepsilon)$

$$(2.9) \qquad \Pr\left(S_{k_1(x,u)} > \log(1 + \delta x^{-1}) - m_{k_1(x,u)}\right)$$

$$\leqq \Pr\left(S_{k_1(x,u)} > \frac{1}{2}\log(1 + \delta x^{-1})\right)$$

(and by the Chebishev inequality)

$$\leqq \left(\frac{1}{2}\log(1 + \delta x^{-1})\right)^{-2} \sum_{j=k_1(x,u)}^{\infty} b_j$$

(and by (2.8))

$$\leqq \left(\frac{1}{2}\log(1 + \delta^2)\right)^{-2} \sum_{j=k_1(x,u)}^{\infty} b_j$$

(and by (1.1) we have for $u > u_1(\varepsilon) \geqq u_0(\varepsilon)$)

$$< \varepsilon \text{ for } u > u_1(\varepsilon), \text{ uniformly in } 0 < x \leqq \delta(\varepsilon)^{-1}.$$

By (2.5), (2.6) and (2.9) we get

$$|I_2(u)| \leqq (2M+2)\varepsilon \quad \text{for } u > u_1(\varepsilon) \quad \text{uniformly in } 0 < x < +\infty. \tag{2.10}$$

The argument leading to (2.10) yields also

$$|I_3(u)| \leqq (2M+2)\varepsilon \quad \text{for } u > u_2(\varepsilon) \quad \text{uniformly in } 0 < x < +\infty. \tag{2.11}$$

Combining (2.3), (2.4), (2.10) and (2.11) we get

$$|P_u(f,x) - f(x)| \leqq \begin{cases} (4M+5)\varepsilon & \text{for } u > u_3(\varepsilon) \text{ and } 0 < x \leqq +\infty \\ \left| f\left(\frac{1}{u}\right) - f(0) \right| & \text{for } u > 0 \text{ and } x = 0 \end{cases}$$

$$\to 0 \text{ as } u \to +\infty, \text{ uniformly in } 0 \leqq x \leqq +\infty.$$

Q.E.D.

In order to prove Theorem 2 we need the following two lemmas.

LEMMA 2. (A variant of Liapounov's theorem.) *Let X_j $(j \geqq 1)$ be independent random variables with finite means m_j and finite variances b_j. Suppose* (1.1)

$$\left(\sum_{k=m+1}^{\infty} E|X_k - m_k|^{2+\delta} \right)^{1/(2+\delta)} = o\left(\left(\sum_{k=m+1}^{\infty} b_k \right)^{1/2} \right), \text{ as } m \to \infty. \tag{2.12}$$

Then

$$\Pr\left\{ S_{m+1} \Big/ \left(\sum_{k=m+1}^{\infty} b_k \right)^{1/2} \leqq y \right\} \to \Phi(y) \tag{2.13}$$

uniformly in y, $-\infty < y < +\infty$, where $\Phi(y)$ is the normal distribution function.

PROOF. The random variable $S_k = \sum_{j=k}^{\infty}(X_j - m_j)$ is an almost surely finite random variable. The proof of (2.13) is similar to that of theorem B (ii) on page 275 of [5] and the uniform convergence follows by theorem 4.3.3 on page 183 of [7].

LEMMA 3. *Let X_j $(j \geqq 1)$ be non-negative independent random variables with finite means m_j and nonzero finite variances b_j. Suppose that for some x_0, $0 < x_0 < +\infty$, and some $\delta > 0$* (1.9) *and* (1.10) *are satisfied. Then*

$$\lim_{u \to +\infty} \sum_{k=n(u)+1}^{\infty} P_k(-\log xu) = \frac{1}{2}. \tag{2.14}$$

PROOF. By the very definition of $n(u)$ we have $u^{-1}\rho_{n(u)} \leqq x_0 < u^{-1}\rho_{n(u)+1}$. This implies $0 \geqq -\log x_0 u + \Sigma_{j=1}^{n(u)} m_j > -m_{n(u)+1}$ and by (1.10) we get

$$\text{(2.15)} \qquad \left(-\log x_0 u + \sum_{j=1}^{n(u)} m_j\right) \Big/ \left(\left(\sum_{j=n(u)+1}^{\infty} b_j\right)^{1/2}\right) \to 0 \text{ as } u \to +\infty.$$

By (2.2)

$$\sum_{k=0}^{n(u)} P_k(-\log x_0 u) = 1 - \Pr\left(S_{n(u)+1} > -\log x_0 u + \sum_{j=1}^{n(u)} m_j\right)$$

$$= 1 - \Pr\left\{S_{n(u)+1} \Big/ \left(\left(\sum_{k=n(u)+1}^{\infty} b_k\right)^{1/2}\right) > \left(-\log x_0 u + \sum_{j=1}^{n(u)} m_j\right) \Big/ \left(\left(\sum_{k=n(u)+1}^{\infty} b_k\right)^{1/2}\right)\right\}.$$

Applying Lemma 2 and (2.15) we get

$$\sum_{k=0}^{n(u)} P_k(-\log x_0 u) \to \Phi(0) = \frac{1}{2}, \quad \text{as} \quad u \to +\infty.$$

Q.E.D.

PROOF OF THEOREM 2. Given $\varepsilon > 0$ we have $f(x) \leqq L^+(x_0) + \varepsilon$ for $x_0 < x < x_0 + \delta(\varepsilon)$ and $f(x) \leqq L^-(x_0) + \varepsilon$ for $x_0 - \delta(\varepsilon) < x < x_0$. So

$$P_u(f, x_0)$$

$$= \left\{ \sum_{\substack{k \\ |u^{-1}\rho_k - x_0| \geqq \delta(\varepsilon)}} + \sum_{\substack{k \\ -\delta(\varepsilon) < u^{-1}\rho_k - x_0 < 0}} + \sum_{\substack{k \\ 0 < u^{-1}\rho_k - x_0 < \delta(\varepsilon)}} + \sum_{\substack{k \\ u^{-1}\rho_k = x_0}} \right\} f(u^{-1}\rho_k) P_k(-\log x_0 u)$$

$$\equiv I_1(u) + I_2(u) + I_3(u) + I_4(u).$$

By applying Theorem 1 to the function $g(x)$ (instead of $f(x)$ there) defined by $g(x) = 0$ for $|x - x_0| < \delta(\varepsilon)$ and $g(x) = f(x)$ for $|x - x_0| \geqq \delta(\varepsilon)$ we get

$$\lim_{u \to \infty} I_1(u) = 0.$$

Since $f(x)$ is bounded and $\lim_{u \to \infty} P_k(-\log x_0 u) = 0$ for $k = 0, 1, 2, \cdots$ we get

$$\lim_{u \to \infty} I_4(u) = 0.$$

Now

$$I_3(u) \leqq (L^+(x_0)+\varepsilon) \sum_{\substack{k \\ u^{-1}\rho_k > x_0}} P_k(-\log x_0 u)$$

$$= (L^+(x_0)+\varepsilon) \sum_{k=n(u)+1}^{\infty} P_k(-\log x_0 u)$$

(and by Lemma 3)

$$\to \frac{1}{2}(L^+(x_0)+\varepsilon).$$

Thus

$$\overline{\lim_{u\to\infty}}\, I_3(u) \leqq \frac{1}{2} L^+(x_0) + \frac{\varepsilon}{2}\,.$$

Similarly we get

$$\overline{\lim_{u\to\infty}}\, I_2(u) \leqq \frac{1}{2} L^-(x_0) + \frac{\varepsilon}{2}\,.$$

Hence

$$\overline{\lim_{u\to\infty}}\, P_u(f, x_0) \leqq \frac{1}{2}(L^+(x_0)+L^-(x_0)).$$

Similarly we get

$$\underline{\lim_{u\to\infty}}\, P_u(f, x_0) \geqq \frac{1}{2}(l^+(x_0)+l^-(x_0)).$$

Q.E.D.

Proof of Theorem 3. X_j are independent exponentially distributed random variables, with means a_j^{-1} and therefore with variances $1/a_j^2$, i.e., their densities are

$$f_{X_j}(t) = \begin{cases} a_j e^{-a_j t} & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t \leqq 0. \end{cases}$$

So the random variable $X_j - (1/a_j)$ has the probability density function

$$h_j(x) \equiv f_{X_j - a_j^{-1}}(x) = \begin{cases} a_j e^{-1} e^{-a_j x} & \text{for} \quad x > -\dfrac{1}{a_j} \\ 0 & \text{for} \quad x \leqq -\dfrac{1}{a_j} \end{cases}$$

As $\Sigma\, 1/a_j^2 < \infty$, $S_k \equiv \Sigma_{j=k}^{\infty}(X_j - (1/a_j))$ is an almost surely finite random variable for $k \geqq 1$. For each $k = 1, 2, \cdots$ we have $S_k = (X_k - (1/a_k)) + S_{k+1}$ where $X_k - (1/a_k)$ and S_{k+1} are two independent random variables and $X_k - (1/a_k)$ is an absolutely continuous random variable. Therefore by a well-known result S_k is absolutely continuous with a probability density function f_k, satisfying

$$(2.16)\qquad \begin{aligned} f_k(x) &= (h_k * f_{k+1})(x) \\ &\equiv \int_{-\infty}^{x+a_k^{-1}} a_k e^{-1} e^{-a_k(x-t)} f_{k+1}(t)\,dt. \end{aligned}$$

Differentiating (2.16) on both sides, we get

$$(2.17)\qquad (d/dx) f_k(x) = a_k (f_{k+1}(x + (1/a_k)) - f_k(x)) \quad \text{for} \quad k = 1, 2, \cdots \quad \text{and} \quad -\infty < x < +\infty,$$

which implies that f_k has derivatives of all orders. Also we get by (2.17) that

$$(2.18)\qquad f_{k+1}(x) = (1 + D/a_k) f_k(x - 1/a_k) \quad \text{for} \quad -\infty < x + \infty \quad \text{and} \quad k = 1, 2, \cdots.$$

By induction we obtain

$$(2.19)\qquad f_{k+1}(x) = \left\{ \prod_{j=1}^{k} \left(1 + \frac{D}{a_j}\right) \right\} f_1\left(x - \sum_{r=1}^{k} \frac{1}{a_r}\right) \quad \text{for} \quad -\infty < x < +\infty \quad \text{and} \quad k = 0, 1, 2, \cdots.$$

By (2.17) we have for $k = 1, 2, \cdots$ and $-\infty < x < +\infty$

$$(2.20)\qquad \begin{aligned} f_k(x) &= \int_{-\infty}^{x} \frac{d}{dt} f_k(t)\,dt \\ &= a_k \left\{ \int_{-\infty}^{x} f_{k+1}\left(t + \frac{1}{a_k}\right) dt - \int_{-\infty}^{x} f_k(t)\,dt \right\} \\ &= a_k \left[\Pr\left(S_{k+1} \leqq x + \frac{1}{a_k}\right) - \Pr(S_k \leqq x) \right] \\ &= a_k \Pr\left(S_{k+1} - \frac{1}{a_k} \leqq x < S_k\right). \end{aligned}$$

By (2.20) and (2.19) we have for $k = 1, 2, \cdots$

$$\Pr\left(S_{k+1} - \frac{1}{a_k} \leqq x < S_k\right) = \frac{1}{a_k} f_k(x) \tag{2.21}$$

$$= \frac{1}{a_k}\left\{\prod_{j=1}^{k-1}\left(1 + \frac{D}{a_j}\right)\right\} f_1(x)$$

$$= \frac{1}{a_k}\left\{D \prod_{j=1}^{k-1}\left(1 + \frac{D}{a_j}\right)\right\} F_1(x).$$

Since S_1 is an almost surely finite random variable, we have

$$\varphi_{S_1}(t) = \prod_{j=1}^{\infty} \varphi_{X_j - a_j^{-1}}(t)$$

$$= 1 \Big/ \prod_{j=1}^{\infty}\left(1 - \frac{it}{a_j}\right) e^{it/a_j}$$

$$= \frac{1}{E(it)}.$$

Since a probability density function is uniquely determined by its characteristic function we get by theorem 6.1, page 55 of [2]

$$\frac{1}{E(s)} = \int_{-\infty}^{+\infty} e^{-st} G(t) dt$$

where $G(t) = f_1(t)$, and the integral converges for $\operatorname{Re} s < \min_{j \geqq 1} a_j$. Also by corollary 5.4, page 53 of [2]

$$f_1(t) \equiv G(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-itx}}{E(ix)} dx.$$

Q.E.D.

PROOF OF THEOREM 5. In this case we clearly have

$$EX_j = m_j = a_j^{-1},$$

$$\operatorname{Var} X_j = b_j = a_j^{-2},$$

$$E\,|X_j - m_j|^3 = 2a_j^{-3},$$

$$\rho_k = \xi_k,$$

and

$$n(u) = k(u).$$

We see that (1.26) is equivalent to (1.9) with $\delta = 1$. Moreover, (1.26) implies (1.10). Indeed we have

$$0 < a_{k(u)+1}^{-1} \leqq \left\{ \sum_{j=k(u)+1}^{\infty} a_j^{-3} \right\}^{1/3}$$

$$= o\left\{ \sum_{j=k(u)+1}^{\infty} a_j^{-2} \right\}^{1/2} \qquad (u \to \infty)$$

so

$$a_{k(u)+1}^{-1} = o\left\{ \sum_{j=k(u)+1}^{\infty} a_j^{-2} \right\}^{1/2}$$

which is exactly (1.10) in our special case of exponentially distributed random variables. It is quite obvious by Theorem 3 that all the other conditions of Theorem 2 are met so the conclusion of Theorem 5 follows.

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